

# Analysis of Maxwell Equations in a Gravitational Field

by

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## Abstract

In a gravitational field, we analyze the Maxwell equations, the corresponding electro-magnetic wave and continuity equations. A particular solution for parallel electric and magnetic fields in a gravitational background is presented. These solutions also satisfy the free-wave equations and the phenomenology suggested by plasma physics.

# 1 Introduction

Plasma and Astrophysical Plasma physicists support the possible existence of electromagnetic stationary waves with parallel  $\vec{E}$  and  $\vec{B}$  fields and consequently having a null Poynting vector [1, 8]. K.R. Brownstein [1] points out that these waves may emerge as solutions of the vector equation  $\vec{\nabla} \times \vec{V} = k\vec{V}$  ( $\vec{V}$  is a vector field and  $k$  is a positive constant). Brownstein considers this equation for the vector potential  $\vec{A}$  as

$$\vec{\nabla} \times \vec{A} = k\vec{A}, \quad (1.1)$$

with a particular solution

$$\vec{A} = a \left[ i \sin kz + \hat{j} \cos kz \right] \cos \omega t \quad (1.2)$$

and takes the associated electric and magnetic field as

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = ka \left[ i \sin kz + \hat{j} \cos ka \right] \sin \omega t \quad (1.3)$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A} = ka \left[ \hat{i} \sin kz + j \cos kz \right] \cos \omega t. \quad (1.4)$$

The Brownstein fields (1.3) and (1.4) satisfy Maxwell equations and the usual vacuum free-wave equation. Moreover, fields,  $\vec{E}$  and  $\vec{B}$  and the vector potential,  $\vec{A}$ , are parallel everywhere. The associated electromagnetic wave has a null Poynting vector and, so, does not propagate energy. The behavior of this wave is just like the phenomenology suggested by Plasma and Astrophysical Plasma Physics.

But it is pointed out [9] that this model is not complete, at least for Astrophysical Plasma. There it is argued that in an Astrophysical Plasma, gravitation must be taken into account in a way that the gravitational background can break the parallelism between

the fields  $\vec{E}$  and  $\vec{B}$ . Consequently, the corresponding electromagnetic wave does not have a null Poynting vector. It is not a stationary wave and propagates energy.

In this work, we consider the electromagnetic and gravitational coupling and we analyze, in a gravitational background, the corresponding Maxwell equations. We get the associated free-wave equation for fields  $\vec{E}$  and  $\vec{B}$  and discuss the corresponding electrostatic regime. That is, the situation in a gravitational background in which the field  $\vec{E}$  does not induce the field  $\vec{B}$  and vice-versa.

And we show that, in a particular situation, electric field  $\vec{E}$  may depend on time but does not induce magnetic field  $\vec{B}$  with or without sources. Finally we show particular electrostatic and magnetostatic solutions for Maxwell equations without sources in a gravitational background. These solutions are such that the electric field,  $\vec{E}_0$  and the magnetic field,  $\vec{B}_0$ , both satisfy the corresponding free-wave equations. Moreover these fields behave as the phenomenology in Plasma Astrophysics has suggested, that is,  $\vec{E}_0$  and  $\vec{B}_0$  are parallel fields. Thus, the associated electromagnetic wave is stationary, has a null vector Poynting and does not propagate energy.

## 2 Electromagnetic and Gravitational Coupling

The action for the gravitational and electromagnetic coupling is written as

$$\mathcal{S} = \int \sqrt{-g} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) d^4s , \quad (2.1)$$

requiring stationary action ( $\delta S = 0$ ) the corresponding Maxwell inhomogeneous equations in a gravitational background are

$$\mathcal{D}_\mu F^{\mu\nu} = \mathcal{J}^\nu , \quad (2.2)$$

where the covariant derivative above is given as:

$$\mathcal{D}_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma_{\mu\lambda}^{\mu\nu} + \Gamma_{\mu\lambda}^\nu \Gamma^{\mu\lambda} = \mathcal{J}^\nu . \quad (2.3)$$

The corresponding homogeneous Maxwell equations are

$$\mathcal{D}_\mu F_{\nu\rho} + \mathcal{D}_\nu F_{\rho\mu} + \mathcal{D}_\rho F_{\mu\nu} = 0 . \quad (2.4)$$

The connection terms of this equation cancel each other in such a way that this equation is the usual homogeneous Maxwell equation

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 . \quad (2.5)$$

Finally, to analyse these equations we adopt the F.R.W. cosmological metric

$$dS^2 = dt^2 - (a(t))^2 \left\{ (1 - Ar^2)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right\} , \quad (2.6)$$

where the term  $a(t)$  is the scale factor and the constant  $A$  may assume values  $A = 1, 0, -1$ . Each value represents the associated curvature of F.R.W. spatial metric.

The second term  $\Gamma_{\mu\lambda}^\mu$  of the covariant derivative equation (2.3) can be written as

$$\Gamma_{\mu\lambda}^\mu = - \frac{\partial h}{\partial x^\lambda} \quad (2.7)$$

where

$$h = \ln \sqrt{-\tilde{g}} \quad (2.8)$$

and  $\tilde{g}$  is the metric determinant. In terms of the electric field,  $E^i = F^{0i}$ , and the magnetic field  $B^i = \varepsilon^{ijk} F_{jk}$ , the Maxwell eq. (2.4) and (2.3) are explicitly given by

$$\vec{\nabla} \cdot \vec{E} = \rho_{(\vec{x})} + \vec{\nabla} h \cdot \vec{E} , \quad (2.9)$$

$$\vec{\nabla} \cdot \vec{B} = 0 , \quad (2.10)$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} , \quad (2.11)$$

$$\vec{\nabla} \times \vec{B} = \vec{J}(\vec{x}) + \frac{\partial \vec{E}}{\partial t} - \frac{\partial h}{\partial t} \vec{E} + \vec{\nabla} h \times \vec{B} . \quad (2.12)$$

Taking the divergence of equation (2.12) and using eq. (2.9) we get the continuity equation in a gravitational background

$$\frac{\partial \rho}{\partial t} - \frac{\partial h}{\partial t} \rho + \vec{\nabla} \cdot \vec{J} - \vec{\nabla} h \cdot \vec{J} = 0 . \quad (2.13)$$

which is expressed in a covariant way as

$$\mathcal{D}_\mu J^\mu = 0 , \quad (2.14)$$

where the covariant derivative is  $\mathcal{D}_\mu = \partial_\mu - \partial_\mu h$ .

Without electromagnetic sources ( $\rho = 0$  ;  $\vec{J} = 0$ ) the free electromagnetic field equations in a gravitational background are:

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} h \cdot \vec{E} , \quad (2.15)$$

$$\vec{\nabla} \cdot \vec{B} = 0 , \quad (2.16)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} , \quad (2.17)$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} - \frac{\partial h}{\partial t} \vec{E} + \vec{\nabla} h \times \vec{B} . \quad (2.18)$$

Taking the curl of equation (2.17), using equations (2.15) and (2.18) and, since  $\vec{\nabla} h$  does not depend explicitly on time, we get the free-wave equation in a gravitational background for the electric field:

$$\nabla^2 \vec{E} - \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{\nabla}(\vec{\nabla} h \cdot \vec{E}) - \frac{\partial \left( \frac{\partial h}{\partial t} \vec{E} \right)}{\partial t} - \vec{\nabla} h \times (\vec{\nabla} \times \vec{E}) . \quad (2.19)$$

Similarly, taking the curl of eq. (2.18), and using eq. (2.16) and, since  $\frac{\partial h}{\partial t}$  does not depend explicitly on the spatial coordinates, we get the free-wave equation in a gravitational background for the magnetic field:

$$\nabla^2 \vec{B} - \frac{\partial^2 \vec{B}}{\partial t^2} = -\vec{\nabla} \times (\vec{\nabla} h \times \vec{B}) - \frac{\partial h}{\partial t} \frac{\partial \vec{B}}{\partial t} . \quad (2.20)$$

### 3 Analysis of Maxwell Equations in a Gravitational Background

Now we analyze the necessary conditions to obtain electrostatic and magnetostatic solutions in the gravitational background, that is, the conditions that the electric field  $\vec{E}$  and the magnetic field  $\vec{B}$  must satisfy in order that the electric field  $\vec{E}$  does not induce the magnetic field  $\vec{B}$  and vice-versa.

From eq. (2.11) it is clear that the gravitational field does not modify the Faraday's law and so it is simple to conclude that the magnetic field  $\vec{B}$  must be stationary in order not to induce an electric field  $\vec{E}$ . On the other hand, the Ampère Law (2.12) has been modified by gravitation. Thus, it is simple to see that even if the electric field is stationary  $\left( \frac{\partial \vec{E}}{\partial t} = 0 \right)$  it may induce a magnetic field  $\vec{B}$  from the gravitation term  $-\frac{\partial h}{\partial t} \vec{E}$ . For example, if we have a non-null stationary electric field  $\left( \frac{\partial \vec{E}}{\partial t} = 0 ; \vec{E} \neq 0 \right)$  and no current density ( $\vec{j} = 0$ ), eq. (2.18) becomes  $\vec{\nabla} \times \vec{B} = -\frac{\partial h}{\partial t} \vec{E} + \vec{\nabla} h \times \vec{B}$  and so it implies that we necessarily have a non-null magnetic field  $\vec{B}$ , since  $\vec{B} = 0$  is not solution for this equation. From eq. (2.12) it is clear that even if the electric field  $\vec{E}$  depends on time it shall not induce a magnetic field  $\vec{B}$  provided the condition below is satisfied:

$$\frac{\partial \vec{E}}{\partial t} - \frac{\partial h}{\partial t} \vec{E} = 0 . \quad (3.1)$$

From condition (3.1) and from eq. (2.9) it is simple to verify that the electric field  $\vec{E}(\vec{r}, t)$  and the charge density  $\rho(t)$  can be written as:

$$\vec{E}(\vec{r}, t) = (a(t))^3 \vec{E}_0(\vec{r}) , \quad (3.2)$$

and

$$\rho(\vec{r}, t) = (a(t))^3 \rho_0(\vec{r}) ; \quad (3.3)$$

where  $\vec{E}_0(\vec{r})$  is a stationary vector field and  $\rho_0(\vec{r})$  is stationary “charge density”. Furthermore the current density  $\vec{J}$  must satisfy the equation:

$$\vec{\nabla} \cdot \vec{J} - \vec{\nabla} h \cdot \vec{J} = 0 \quad (3.4)$$

If the conditions (3.1)-(3.4) are verified and if the magnetic field  $\vec{B}$  is static, the electrostatic and magnetostatic field-equations become

$$\vec{\nabla} \cdot \vec{B} = 0 , \quad (3.5)$$

$$\vec{\nabla} \times \vec{B} = \vec{j} + \vec{\nabla} h \times \vec{B} , \quad (3.6)$$

$$\vec{\nabla} \cdot \vec{E}_0 = \rho_0(\vec{r}) + \vec{\nabla} h \cdot \vec{E}_0 , \quad (3.7)$$

$$\vec{\nabla} \times \vec{E}_0 = 0 . \quad (3.8)$$

Any solution of equation (3.4)-(3.6) for the charge current density  $\vec{J}(\vec{r}, t)$  and for the magnetic field,  $B(\vec{r}, t)$ , can be combined with any solution of equations (3.2), (3.3), (3.7) and (3.8). So under these conditions the magnetic field,  $B(\vec{r}, t)$ , does not induce the electric field,  $E(\vec{r}, t)$ , and vice-versa. This way we call these equations magnetostatic and electrostatic, but we must remember the electric field is not static but depends on time according to eq. (3.2).

## 4 Solutions for Maxwell Equations without Sources in a Gravitational Scenery

We now consider the Maxwell equations (2.9)-(2.12) without sources ( $\rho = 0$  ;  $\vec{J} = 0$ ) with the condition that the electric field  $\vec{E}(\vec{r}, t)$  satisfies eq. (3.1) and that the magnetic field  $\vec{B}$  does not depend explicitly on time. The Maxwell equations for the electrostatic and magnetostatic field-equations without sources in a gravitational field (3.5)-(3.8) are given by:

$$\vec{\nabla} \cdot \vec{B} = 0 , \quad (4.1)$$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} h \times \vec{B} , \quad (4.2)$$

$$\vec{\nabla} \cdot \vec{E}_0 = \vec{\nabla} h \cdot \vec{E}_0 , \quad (4.3)$$

$$\vec{\nabla} \times \vec{E}_0 = 0 . \quad (4.4)$$

A particular solution in spherical coordinates is

$$E_0(\vec{r}, t) = \frac{\alpha}{r \sin \theta} \implies \vec{E}(\vec{r}, t) = \alpha \frac{(a(t))^3}{r \sin \theta} \hat{\varphi} \quad (4.5)$$

and

$$\vec{B}(\vec{r}, t) = \beta \frac{r}{\sqrt{1 - Ar^2}} \hat{\varphi} \quad (4.6)$$

where  $\alpha$  and  $\beta$  are constants and  $\hat{\varphi}$  is the unit azimuthal vector:  $\hat{\varphi} = (-\hat{i} \sin \varphi + \hat{j} \cos \varphi)$ .

A simple substitution show that these fields satisfy all the Maxwell equations without sources (2.15)-(2.18). It is interesting to point out that the electric and magnetic fields are parallel and satisfy the free-wave eq. (2.19) and eq. (2.20). The electromagnetic wave has



parallel electric and magnetic fields, has a null Poynting vector, it is a stationary wave and it does not propagate energy as suggested by the Astrophysical Plasma phenomenology.

The term  $\vec{\nabla}h$  and the unitary azimuthal vector  $\hat{\varphi}$  are perpendicular vectors, so  $(\hat{\varphi}, \vec{\nabla}h, \hat{\varphi} \times \vec{\nabla}h)$  form a complete set and any vector can be written as:

$$\vec{V} = V_{\varphi}\hat{\varphi} + V_h\vec{\nabla}h + V_{\varphi h}(\hat{\varphi} \times \vec{\nabla}h) \quad (4.7)$$

The vector components  $V_{\varphi}$ ,  $V_h$  and  $V_{\varphi h}$  may depend on time and spatial coordinates. General solutions  $E(\vec{r}, t)$  and  $\vec{B}(\vec{r}, t)$  for Maxwell equations in a gravitational background using the ansatz (4.7) are being considered.

We conclude that in astrophysical plasma it is important to consider electromagnetic and gravitational coupling and this coupling modifies the free-wave equations for electric and magnetic fields.

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